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HOMOGENEOUS SOLUTIONS FOR A PRESTRESSED ELASTIC PLATE

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Symbolic methods [1, 2] are used to construct homogeneous solutions for the problem of bending of a plate previously deformed in its plane. A model of neo—Hookean material is employed, and this is found useful in describing resin-like materials. Inspection of the characteristic equation shows that a preliminary application of a load changes substantially the character of the spectrum of the homogeneous solutions. Analysis of the penetrating solution which is an analog of the Lur'e's biharmonic solution in the theory of unstressed plates, attests the validity of the Kirchhoff's kinematic hypotheses for a thin plate and of the classical Saint Venant's equation of stability of plates under the condition of small initial stresses.

The homogeneous solutions obtained can be used, with the help of the asymptotic method [3] to investigate, in particular, the problems of stability of thick plates when the conditions specified at the side surface are arbitrary.

1. Constructing the homogeneous solutions. Let us consider a plate made of an incompressible neo —Hookean material subjected to an initial deformation of the form

$$y_1 = \lambda x_1, \quad y_2 = \lambda x_2, \quad y_3 = \lambda^{-2} x_3 \tag{1.1}$$

where λ is a constant and x_k, y_k $(k = 1, 2, 3)$ are Cartesian coordinates before and after the dermormation respectively. Such a deformation is realised in a plate of arbitrary form, in its plane, when its side surface is under a uniform load. A small bending deformation is superimposed on this deformation, and the bending deformation is described by the following equations [4]:

$$\begin{aligned} & D^2 u_i + u_i'' + \lambda^{-1} \partial_i p = 0 \quad (i = 1, 2) \\ & D^2 w + w'' + \lambda^2 p' = 0 \\ & \partial_1 u_1 + \partial_2 u_2 + \lambda^3 w' = 0, \quad D^2 = \partial_1^2 + d_2^2 \end{aligned} \tag{1.2}$$

Here u_1, u_2 and w are the components of the vector of additional translational displacements in the $x_1, x_2, x_3 = z$ coordinate system; a prime denotes differentiation with respect to z; ∂_i $(i = 1, 2)$ is the differential operator in x_i and p is an unknown function of the coordinates appearing as a result of the incompressibility of the material. The last equation of (1.2) expresses the condition of incompressibility.

The boundary conditions at the faces $z = \pm h$ of the plate express the absence of an additional load and have the form [4]

$$u_i' + \lambda^{-3}\partial_i w = 0 \quad (i = 1, 2), \quad 2w' + \lambda^2 p = 0 \tag{1.3}$$

In deriving (1.2) and (1.3) it was assumed that in the initial homogeneous deformed state the stress $\sigma_z = 0$. The stress acting in the plane of the plate is given in terms of the coefficient λ by the formula (G is the shear modulus of the material)

$$\sigma = G(\lambda^2 - \lambda^{-4}) \tag{1.4}$$

Integrating the system (1.2) with boundary conditions (1.3) with help of the symbolic method [1, 2], we obtain the following expressions for the displacement components and the function p:

$$u_i = \sum_{k=1}^{3} A_{ik}\chi_k \quad (i = 1, 2), \quad w = \sum_{k=1}^{3} B_k\chi_k, \quad p = \sum_{k=1}^{3} C_k\chi_k \tag{1.5}$$

Here A_{ik}, B_k, C_k $(i = 1, 2;\ k = 1, 2, 3)$ are the differential operators of infinite order in x_1, x_2 with constant coefficients, and χ_1, χ_2, χ_3 denote an arbitrary triad of solutions of the equation $Q\chi = 0$ where Q is the operator determinant of the problem (1.2), (1.3). The operators Q, A_{ik}, B_k and C_k are given by the following expressions $(\gamma \equiv \lambda^{-3})$:

$$\begin{aligned}
&Q = hD^2Q_1 \cos hD\\
&A_{11} = hz\,(D^2\Delta_{13}Q_1 + \gamma_1\gamma^2\partial_1{}^2T), \quad A_{12} = hz\gamma_1\gamma^2\partial_1\partial_2T\\
&A_{13} = z\gamma_1\gamma\partial_1\Delta_{12}\,(\gamma_2\Delta_{12}\Delta_{23} - 2\Delta_{22}\Delta_{13})\\
&B_1 = h\gamma_1\gamma\partial_1\Delta_{12}\,(2\Delta_{11}\Delta_{24} - \gamma_2\Delta_{21}\Delta_{14})\\
&B_3 = \gamma_1\Delta_{12}\,(\gamma_2\Delta_{12}\Delta_{24} - 2\gamma^2\Delta_{22}\Delta_{14})\\
&C_1 = -2hz\gamma^2\lambda\partial_1D^2\Delta_{11}\Delta_{12}\Delta_{23},\ C_3 = -z\gamma\lambda\gamma_2D^2\Delta^2_{12}\Delta_{23}\\
&Q_1 = -\gamma_1\,(\gamma_2{}^2\Delta_{12}\Delta_{21} - 4\gamma^2\Delta_{22}\Delta_{11}), \quad \gamma_1 = (1 - \gamma^2)^{-1}\\
&\gamma_2 = 1 + \gamma^2, \quad T = 2\Delta_{11}\,(\Delta_{12}\Delta_{23} - 2\Delta_{22}\Delta_{13}) + \gamma_2\Delta_{12}\Delta_{21}\Delta_{13}\\
&\Delta_{11} = P\,(hD, \pi / 2), \quad \Delta_{12} = P\,(\pi, hD), \quad \Delta_{13} = P\,(zD, \pi / 2)\\
&\Delta_{14} = P\,(\pi, zD), \quad \Delta_{21} = P\,(h\gamma D, \pi / 2), \quad \Delta_{22} = P\,(\pi, h\gamma D)\\
&\Delta_{23} = P\,(z\gamma D, \pi / 2), \quad \Delta_{24} = P\,(\pi, z\gamma D)\\
&P\,(x, y) = x^{-1}\sin x + \cos y
\end{aligned}$$

and A_{2k}, B_2 and C_2 can be obtained from A_{1k}, B_1 and C_1 by making the substitution $\partial_1 \sim \partial_2$.

We note that when $\gamma \to 1$, i.e. when the initial deformation is removed, the operator Q becomes an operator determinant of the theory of bending of an unstressed plate [1] (where Poisson's ratio should be made equal to $^1/_2$ since the material is incompressible).

We seek the solution of the equation $Q\chi = 0$, as in [1], in the class of functions satisfying the Helmholtz equation

$$D^2\chi - \frac{\alpha^2}{h^2}\chi = 0$$

We use the following equation to determine the corresponding values of the parameter α :

$$\frac{\alpha^2 \cos\alpha}{\gamma^2 - 1}\left[(1+\gamma^2)^2 \cos\alpha \frac{\sin\alpha\gamma}{\alpha\gamma} - 4\gamma^2 \cos\alpha\gamma \frac{\sin\alpha}{\alpha}\right] = 0 \tag{1.6}$$

and, in what follows, we shall call it the characteristic equation. In the limit as $\gamma \to 1$, this equation becomes the known equation [1]

$$2\alpha^2 \cos\alpha \left(1 - \frac{\sin 2\alpha}{2\alpha}\right) = 0 \tag{1.7}$$

Obviously the zeros of the cosine $\sigma_t = \pi(2t-1)/2,\ t = 0, \pm 1, \pm 2, \ldots$ ..., are roots of both (1.7) and (1.6). However, if (1.7) has a quadruple zero root, then for (1.6) this root will only be a double root. Below we shall show that equation (1.6) has two nonzero roots differing from each other only by the sign, both tending to zero as $\gamma \to 1$, both real when $0 < \gamma < 1$ (pretensioned plate), and both pure imaginary when $1 < \gamma < \gamma_* \approx 3.383$ (precompressed plate). The value $\gamma = \gamma_*$ is a singularity, and the roots tend to infinity along the imaginary axis as $\gamma \to \gamma_*$. Let us denote the roots belonging to the closure of the first quadrant of the complex plane by α_0. Then the other root will be $-\alpha_0$. We denote the remaining roots of (1.6) by α_q, $q = 1, 2, 3, \ldots$ (the method of numbering to be given later). Since the roots are distributed symmetrically over the complex plane, it is sufficient to consider the solutions of (1.6) found in the first quadrant. Since there are three groups of roots of the characteristic equation, we can construct in the problem under consideration, three types of homogeneous solutions.

T h e p e n e t r a t i n g s o l u t i o n. Assuming in (1.5) $\chi_1 = \chi_2 = 0$, $\chi_3 = -\psi$, where the function ψ satisfies the equation

$$D^4\psi - \frac{\alpha_0^2}{h^2} D^2\psi = 0 \tag{1.8}$$

we obtain the following representation for the penetrating solution:

$$\begin{aligned}
&u_i = h\gamma\zeta\partial_i\psi + h^3 A(\zeta)\,\partial_i D^2\psi \quad (i = 1, 2) \qquad (1.9)\\
&w = -\psi + h^2 B(\zeta)\, D^2\psi, \quad p = hC(\zeta)\, D^2\psi\\
&A(\zeta) = \alpha_0^{-1} M\,[(1+\gamma^2)\cos\alpha_0 \sin\alpha_0\gamma\zeta -\\
&\quad 2\gamma\cos\alpha_0\gamma \sin\alpha_0\zeta] - \gamma\alpha_0^{-2}\zeta\\
&B(\zeta) = M\,[(1+\gamma^2)\cos\alpha_0 \cos\alpha_0\gamma\zeta - 2\gamma^2 \cos\alpha_0\gamma \cos\alpha_0\zeta] + \alpha_0^{-2}\\
&C(\zeta) = \lambda\alpha^{-1}_0 (1+\gamma^2)\cos^2\alpha_0 \sin\alpha_0\gamma\zeta, \quad M = \alpha_0^{-2}(\gamma^2 - 1)^{-1}\cos\alpha_0
\end{aligned}$$

Here $\zeta = z / h$ is a dimensionless transverse coordinate. We note that when $\gamma \to 1$, the equation (1.8) becomes biharmonic.

The vortical solution. If the function $B_t(x_1, x_2)$ satisfies the equation

$$D^2 B_t - \frac{\sigma_t^2}{h^2} B_t = 0, \quad t = 1, 2, 3, \ldots$$

then, using (1.5) we can write the vortical solution in the form

$$u_1 = h^2 \sum_{t=1}^{\infty} F_t(\zeta)\, \partial_2 B_t, \quad u_2 = -h^2 \sum_{t=1}^{\infty} F_t(\zeta)\, \partial_1 B_t, \quad w = p = 0 \qquad (1.10)$$

$$F_t(\zeta) = 4\gamma^2 (-1)^{t+1} (1 - \gamma^2)^{-1} \sigma_t^{-2} \cos \sigma_t \gamma \sin \sigma_t \zeta$$

Potential solution. Assuming in (1.5) $\chi_3 = -C_q / \cos \alpha_q$, $\chi_1 = \chi_2 = 0$ where the function $C_q(x_1, x_2)$ satisfies the equation

$$D^2 C_q - \frac{\alpha_q^2}{h^2} C_q = 0, \quad q = 1, 2, 3, \ldots$$

we obtain the following representation for the potential solution:

$$u_i = h \sum_{q=1}^{\infty} H_q(\zeta)\, \partial_i C_q \quad (i = 1, 2) \qquad (1.11)$$

$$w = -h \sum_{q=1}^{\infty} M_q(\zeta)\, C_q, \quad p = h^{-1} \sum_{q=1}^{\infty} N_q(\zeta)\, C_q$$

$$H_q(\zeta) = \gamma_1 \alpha_q^{-1} [(1 + \gamma^2) \cos \alpha_q \sin \alpha_q \gamma \zeta - 2\gamma \cos \alpha_q \gamma \sin \alpha_q \zeta]$$

$$M_q(\zeta) = \gamma_1 [(1 + \gamma^2) \cos \alpha_q \cos \alpha_q \gamma \zeta - 2\gamma^2 \cos \alpha_q \gamma \cos \alpha_q \zeta]$$

$$N_q(\zeta) = \lambda \alpha_q (1 + \gamma^2) \cos \alpha_q \sin \alpha_q \gamma \zeta, \quad \gamma_1 = (1 - \gamma^2)^{-1}$$

Below we shall show that the set of all potential solutions (1.11) can be separated into two subsets. The solutions belonging to the first subset have no analogs in the theory of unstressed plates, while the following assertion holds for the second subset as well as for the penetrating (1.9) and vortical (1.10) solutions: when $\gamma \to 1$, the above solutions become the potential, biharmonic and vortical solutions, respectively, of the theory of unstressed plates, with the Poisson's ratio equal to $1/2$ [1].

2. Investigation of the characteristic equation. We reduce the characteristic equation (1.6) to the form (2.1) or (2.2) (neglecting the factor $\alpha^2 \cos \alpha$)

$$P_1(\gamma) \sin \alpha (\gamma + 1) + P_2(\gamma) \sin \alpha (\gamma - 1) = 0 \qquad (2.1)$$

$$P_1(\gamma) = (\gamma^3 - 3\gamma^2 - \gamma - 1) / (2\gamma^2 + 2\gamma), \quad P_2(\gamma) = P_1(-\gamma)$$

$$P_1(\gamma) [e^{i\alpha(\gamma+1)} - e^{-i\alpha(\gamma+1)}] + P_2(\gamma) [e^{i\alpha(\gamma-1)} - e^{-i\alpha(\gamma-1)}] = 0 \qquad (2.2)$$

Since the value $\gamma \leqslant 0$ has no physical sense, it is sufficient to consider the equation (2.1) for $\gamma > 0$. The exponential multiplier in the left hand side of (2.2) is an almost periodic function with a bounded spectrum [5]. Hence the set of roots of (2.2) has the following properties:

1) for every fixed $\gamma \neq 1$ all roots lie in the strip $|\operatorname{Im} \alpha| \leqslant C_\gamma$, $C_\gamma = \text{const}$;
2) for every $\gamma \neq 1$ the set of roots forms an almost periodic point set;
3) the following representation holds for the roots;

$$\alpha_k = \pi k (1 + \gamma)^{-1} + \Psi_\gamma (k), \ k = 0, \pm 1, \pm 2, \ldots \tag{2.3}$$

where $\Psi_\gamma (k)$ is a bounded function assuming complex values.

It can be confirmed that when $\gamma = r / s$ is rational, then all solutions of (2.2) can be written in the form

$$\begin{gathered} \alpha_{(r+s)k+m} = -{}^1/_2 si \ln | t_m | + {}^1/_2 s \arg t_m + \pi ks \\ m = 0, 1, 2, \ldots, r + s - 1; \quad k = 0, \pm 1, \pm 2, \ldots \end{gathered} \tag{2.4}$$

where t_m are solutions of the equation

$$P_1 (r / s) t^{r+s} + P_2 (r / s) t^r - P_2 (r / s) t^s - P_1 (r / s) = 0 \tag{2.5}$$

Thus, in the case of rational γ the set of solutions of (2.2) separates into a finite number of series. It was established in [7] that this also occurs when γ are irrational, and from the geometrical point of view it makes more sense to carry out the numbering according to the series, as in (2.4), than in the order or increasing real parts as in (2.3).

T h e o r e m 1. When $\gamma \neq 1$, Eq. (2.1) has an enumerable set of reat roots.

P r o o f. When γ is rational, the assertion is obvious, since equation (2.5) has a root $t = 1$ for any r and s. If γ is irrational, then it is sufficient to consider the function $f (\gamma, \alpha) = P_1 (\gamma) \sin \alpha (\gamma + 1) + P_2 (\gamma) \sin \alpha (\gamma - 1)$ on the sequence of points of the real axis $\alpha_k = \pi k (\gamma - 1)^{-1}$, $k = 1, 2, 3\ldots$. We have $f (\gamma, \alpha_k) = P_1 (\gamma) (-1)^k \sin \cdot [2\pi k (\gamma - 1)^{-1}]$. Since γ is irrational, the quantity $2\pi (\gamma - 1)^{-1}$ cannot coincide with any of the sine periods. Therefore it can be shown that the change of sign in the sequence $f (\gamma, \alpha_k)$ takes place an even number of times. Since the function $f (\gamma, \alpha)$ is continuous, it has an enumerable set of real zeros.

The value γ_* mentioned above represents a unique real zero of the function $P_1 (\gamma)$.

T h e o r e m 2. If $0 < \gamma < 1$ (case of pretension) or $\gamma > \gamma_*$ (case of a strong, or a very strong precompression), then equation (2.1) has no purely imaginary roots. If $1 < \gamma < \gamma_*$ (case of a moderate precompression), then (2.1) has two purely imaginary roots differing only in sign.

P r o o f. The problem of whether (2.1) has purely imaginary roots can be reduced to that of elucidating the existence of real roots of the equation

$$\varphi(x, \gamma) = \Phi(\gamma) \tag{2.6}$$
$$\varphi(x, \gamma) = \operatorname{th}(\gamma x)\operatorname{th}^{-1}x, \quad \Phi(\gamma) = 4\gamma^3(1+\gamma^2)^{-2}$$

We have $\varphi_x' = (\gamma \operatorname{sh} 2x - \operatorname{sh} 2\gamma x) / (2\operatorname{ch}^2 \gamma x \operatorname{sh}^2 x)$.

It can be established that $\varphi_x'(x, \gamma) < 0$ when $\gamma > 1$ and $\varphi_x'(x, \gamma) > 0$ when $\gamma < 1$. Consequently, if $\gamma > 1$, then $\varphi(x, \gamma)$ decreases monotonously from γ to 1 as x increases from zero to ∞, and increases monotonously from γ to 1 when $\gamma < 1$. It can further be shown that for every $\gamma > 0$ the inequality $\Phi(\gamma) \leqslant \gamma$ holds (the equal sign applies when $\gamma = 1$). Since $\varphi(x, \gamma) > \gamma$ when $\gamma < 1$ and $x > 0$, and $\Phi(\gamma) < \gamma$, the equation (2.6) has no positive zeros when $\gamma < 1$ (and by virtue of the symmetry of $\varphi(x, \gamma)$ in x, it has no negative zeros either). Moreover, computing the derivative $\Phi'(\gamma)$ we can confirm that $\Phi(\gamma)$ increases monotonously in the interval $0 < \gamma < \sqrt{3}$ and decreases monotonously in the interval $\sqrt{3} < \gamma < \infty$. The equation $\Phi(\gamma) = 1$ has two roots, $\gamma = 1$ and $\gamma = \gamma_*$. Since $\varphi(x, \gamma) > 1$ when $\gamma > 1$, therefore (2.6) has no real roots when $\gamma > \gamma_*$. For $1 < \gamma < \gamma_*$ we have $1 < \Phi(\gamma) < \gamma$. Since the function $\varphi(x, \gamma)$ decreases monotonously from γ to 1, the equation (2.6) has a single positive root. This implies that (2.1) has, for $1 < \gamma < \gamma_*$, exactly two purely imaginary roots differing only in sign.

C o r o l l a r y. If $1 < \gamma < \gamma_*$, then Eq. (2.1) has an enumerable set of complex roots. The proof follows from the exitence of purely imaginary roots and the property (2) of the set of roots of (2.1).

We note that the appearance of purely imaginary roots of the characteristic equation when the plate is under initial compression, is not accidental. Since the imaginary roots generate homogeneous solutions oscillating with respect to the coordinates x_1 and x_2 and they do not decay, if follows that the plate compressed in its plane can lose its stability through bending. The special value $\lambda_* = \gamma_*^{-1/2}$ of the initial deformation parameter appearing in the previous discussions coincides with the magnitude of the initial compression under which an arbitrarily thick plate with a sliding clamp along the side surface will lose its stability [4].

We shall call the value γ_0 of the parameter γ the multiple point of (2.1), provided that (2.1) has repeated roots at $\gamma = \gamma_0$.

T h e o r e m 3. If γ is arbitrary, Eq. (2.1) has no roots of multiplicity higher than two. If $\gamma > 1$, then (2.1) has no double roots either. The set of multiple points of (2.1) is an enumerable subset of the segment [0, 1], is dense everwhere and coincides with the set of all zeros of the functions $\Psi_{m,k}(\gamma)$

$$\Psi_{m,k}(\gamma) = (\gamma - 1)\arccos q_1 + (\gamma + 1)\arccos q_2 + 2\pi(k\gamma - m)$$
$$q_1 = -\frac{\gamma^2+1}{(\gamma+1)P_1(\gamma)}, \quad q_2 = \frac{\gamma^2+1}{(\gamma-1)P_2(\gamma)}, \quad |k| > |m|, \quad km > 0$$

where k and m are integers, on the segment [0, 1]. Every function $\Psi_{m,k}(\gamma)$ is monotonous on this segment and has on it a unique zero $\gamma_{m,k}$ for which the following inequality holds:

$$\frac{4|m|-3}{4|k|} \leqslant \gamma_{m,k} \leqslant \frac{4|m|+3}{4|k|}$$

Every multiple point $\gamma_{m,k}$ has a corresponding pair of real double roots $\pm\alpha_{m,k}$, $\alpha_{m,k} > 0$. Using the perturbation theory [8] we can obtain a solution of (2.1) near the point $(\gamma_{m,k}; \alpha_{m,k})$

$$\alpha = \alpha_{m,k} \pm c\delta^{1/2}, \quad \delta = \gamma - \gamma_{m,k}$$
$$\alpha_{m,k} = (1 + \gamma_{m,k})^{-1} \operatorname{sign}(k)\, [\pi (k + m) + \arccos q_1 (\gamma_{m,k})]$$
$$c = \{2 (1 - \gamma_{m,k}^2)^{-1} [\alpha_{m,k} L_1 (\gamma_{m,k}) \operatorname{sign}(k) + L_2 (\gamma_{m,k})]\}^{1/2}$$
$$L_1(\gamma) = \frac{1+\gamma^2}{(1+2\gamma^2-3\gamma^4)^{1/2}}, \quad L_2(\gamma) = -\frac{2\gamma(\gamma^4-2\gamma^2-3)}{\gamma^6-11\gamma^4-5\gamma^2-1}$$

The proof is omitted.

In studying the asymptotic behavior of the roots of (2.1) at the points $\gamma = 1$, $\gamma = \gamma_*$, $\gamma = 0$, $\gamma = \infty$, it is expedient to adopt a method of numbering the roots different from that used in (2.3) or (2.4). We have the following asymptotic formulas. For the case $\gamma \to 1$ we have

$$\alpha_{2k-1}(\gamma) = \pi k (\gamma - 1)^{-1} + \varepsilon_k(\gamma), \quad k = 1, 2, 3, \ldots \tag{2.7}$$

where $\varepsilon_k(\gamma)$ is a bounded function, without a limit as $\gamma \to 1$. The potential solutions corresponding to the roots α_{2k-1} have no analogs in the theory of unstressed plates, and they have no limit values when $\gamma \to 1$.

Using perturbation theory [8] we obtain the following formulas:

$$\alpha_0 = (1-\gamma^2)^{1/2} \sum_{k=0}^{\infty} A_k (1-\gamma^2)^k \tag{2.8}$$
$$A_0 = \sqrt{3}/2, \quad A_1 = 13\sqrt{3}/80, \quad A_2 = 4241\sqrt{3}/44800, \ \ldots$$
$$\alpha_{2k} = \sum_{n=0}^{\infty} A_{kn} (\gamma^2 - 1)^n, \quad k = 1, 2, 3, \ldots \tag{2.9}$$
$$A_{k1} = -\theta_k A_{k0}/4, \quad A_{k2} = \theta_k^{-1} A_{k0} (12 + 19 t_k^2 + 6 t_k^4)/48$$
$$A_{k3} = \theta_k^{-2} A_{k0} (t_k^{10} + 5t_k^8 - 5t_k^6 - 53 t_k^4 - 71 t_k^2 - 27)/192$$
$$\theta_k = 1 + t_k^2$$

Here A_{k0} are the roots of the equation $\sin 2\alpha = 2\alpha$, $t_k = \operatorname{ctg} A_{k0}$. We can find A_{k0} using the asymptotic formula given in [9]. Moreover we can show that the following recurrence relations hold for the coefficients A_k and A_{kn}:

$$A_{k-1} = \frac{3}{8} A_0^{-1} \Big[-\frac{4}{3} \sum_{n=1}^{k-2} A_n A_{k-n-1} + \frac{1}{6} \sum_{n=0}^{k-2} A_n (4A_{k-n-2} + A_{k-n-3}) -$$

$$\sum_{n=2}^{k} b_n \sum_{r=0}^{n} (-1)^r B_{k-n-r,n} C_{n,r}\Big], \quad k \geqslant 3$$

$$A_{kn} = \beta_k \sum_{s=0}^{\infty} b_s \Big[B_{n,s}^{k*} C_{s,0} + \sum_{r=1}^{n} B_{n-r,s}^{k} C_{n,r} \Big], \quad n \geqslant 1, \quad k \geqslant 1$$

$$C_{n,r} = \frac{1}{r!} \frac{d^r}{dt^r} [P_1(\sqrt{t})(\sqrt{t}+1)^{2n+1} + P_2(\sqrt{t})(\sqrt{t}-1)^{2n+1}]|_{t=0}$$

$$b_n = \frac{(-1)^n}{(2n+1)!}, \quad B_{s,n} = \sum_{i_1+i_2+\ldots+i_{2n}=s} A_{i_1} A_{i_2} \ldots A_{i_{2n}}$$

$$\beta_k = -A_{k0}(4\sin^2 A_{k0})^{-1}, \quad B_{n,s}^{k} = \sum_{i_1+i_2+\ldots+i_{2s}=n} A_{ki_1} A_{ki_2} \ldots A_{ki_{2s}}$$

$$B_{n,s}^{k*} = \sum_{i_1+i_2+\ldots+i_{2s}=n;\ i_r \neq n,\ r=1,2,\ldots 2s} A_{ki_1} A_{ki_2} \ldots A_{ki_{2s}}$$

We note that $\alpha_0 \to 0$ as $\gamma \to 1$, the roots α_0 is real when $\gamma < 1$ and purely imaginary when $\gamma > 1$. Further, $\lim \alpha_{2k} = A_{k0}$ as $\gamma \to 1$, i.e. the potential solutions corresponding to the roots α_{2k} become, as $\gamma \to 1$, the potential solutions of the theory of unstressed plates.

Numerical analysis shows that the error given by the formula (2.8) when three coefficients are taken into account on the segment $0.55 \leqslant \gamma \leqslant 1.35$, does not exceed 0.5%. For $k = 1,2$ the formulas (2.9) (also with only three coefficients taken into account) gives an error not exceeding, at $0.8 \leqslant \gamma \leqslant 1.2$, 0.5% for the real part and 3% for the imaginary part.

Case $\gamma \to \gamma_*$

$$\alpha_{2k-1} = \omega(\pi k - q \sin 2\pi k \omega) + o(\gamma - \gamma_*)^2, \ \omega = (\gamma - 1)^{-1} \tag{2.10}$$

$k = 1, 2, 3, \ldots, \ldots$

$$\alpha_{2k} = \pi k - {}^1/_2 i (\ln|q| - \beta e^{2i\pi k\gamma}) + O(|\gamma - \gamma_*|^{2\gamma-2}), \ \gamma < \gamma_* \tag{2.11}$$

$$\alpha_{2k} = \pi(k + {}^1/_2) - {}^1/_2 i (\ln|q| - \beta e^{i\pi(2k+1)\gamma}) + O(|\gamma - \gamma_*|^{2\gamma-2})$$

$\gamma > \gamma_*$

$\beta = (q^2 - 1)|q|^{\gamma-1}, \ k = 0, 1, 2, \ldots$

Case $\gamma \to 0$

$$\alpha_{2k-1} = \gamma^{-1}[\pi k + 0.5(1 - q)] + O(\gamma^4), \ k = 1, 2, 3, \ldots \tag{2.12}$$

$$\alpha_{2k} = \pi(k + {}^1/_2) + (q - 1)[(2k+1)\pi\gamma]^{-1} + O(\gamma^3), \ k = 0, 1, 2, \ldots \tag{2.13}$$

Case $\gamma \to \infty$

$$\alpha_{2k-1} = \pi k \gamma^{-2}(\gamma + 1 - q) + O(\gamma^{-4}), \ k = 1, 2, 3, \ldots \tag{2.14}$$

$$\alpha_{2k} = \pi(k + {}^1/_2) + \gamma^{-1}\mu_k(\gamma), \ k = 0, 1, 2, \ldots \tag{2.15}$$

where $\mu_k(\gamma)$ is a bounded complex function and $\mu_k(\gamma) \to 0$ when $\gamma \to \infty$. In the formulas (2.10) –(2.15) we have $q = P_1(\gamma) / P_2(\gamma)$.

The asymptotic formulas (2.10) –(2.15) give good accuracy and can be used as starting approximations to obtain more accurate values for the roots. For example, for $k = 1$ the greatest error given by the formula (2.11) for $\operatorname{Re}\alpha_2$ and $\operatorname{Im}\alpha_2$ respectively is 1% and 2% for $1.5 \leqslant \gamma \leqslant 2$, 0.3% and 0.5% for $2 \leqslant \gamma \leqslant 2.3$, 0.1% and 0.1% for $2.3 \leqslant \gamma \leqslant 4$. Moreover, as we see from (2.11), when $\gamma = \gamma_*$ the quantity $\operatorname{Re}\alpha_{2k}$ undergoes a jump of $\pi / 2$ and $\operatorname{Im}\alpha_{2k} \to \infty$ as $\gamma \to \gamma_*$.

We note that, as the parameter γ changes continously from 1 to γ_* and from ∞ to γ_*, the roots α_k of the formulas (2.7) –(2.9) and (2.14), (2.15) respectively become the roots α_k of the formulas (2.10), (2.11). The same is true for the root α_0 in (2.8) and (2.13) when γ changes from zero to one. It is not however true for the roots $\alpha_k, k \neq 0$ of the formulas (2.7), (2.9) and (2.12), (2.13), since the latter have multiple points which are the branch points of the spectral curves of (2.1).

3. **Analysis of the penetrating solution.** As we know, in the theory of bending of unstressed plates the components of the biharmonic solution and the stresses defined by these solutions are expressed in terms of the corresponding flexure of the middle plane, the flexure being a biharmonic function. In the present problem the components of the penetrating solution can also be expressed in terms of the corresponding flexure of the middle plane which satisfies the equation (1.8). Indeed, (1.5) yields the following representation for the flexure w_0 of the middle plane corresponding to the penetrating solution:

$$\begin{aligned} w_0 = {} & h\gamma\chi_* + h^3\alpha_0^{-1}\gamma M\,[2 \sin \alpha_0 - (1 + \gamma^2)\,\gamma^{-1} \sin \alpha_0\gamma - \\ & (\alpha_0 M)^{-1}]D^2\chi_* + \chi_3 + \\ & h^2 M\,[(1 + \gamma^2) \cos \alpha_0 - 2\gamma^2 \cos \alpha_0\gamma - \alpha_0^{-2}M^{-1}]\cdot D^2\chi_3 \\ \chi_* = {} & \partial_1\chi_1 + \partial_2\chi_2, \quad M = \alpha_0^{-2}\,(1 - \gamma^2)^{-1} \cos \alpha_0 \end{aligned} \tag{3.1}$$

where the functions χ_k, $k = 1, 2, 3$ are arbitrary solutions of (1.8). Let us put in (3.1)

$$\chi_1 = b\partial_1\chi, \quad \chi_2 = b\partial_2\chi, \quad \chi_3 = \chi + gD^2\chi$$
$$b = \frac{h(1 - \gamma^4)}{2\alpha_0\gamma \sin \alpha_0}\,[(1 + \gamma^2) \cos \alpha_0 - 2\gamma^2 \cos \alpha_0\gamma]^{-1}, \quad g = -\frac{h^2}{\alpha_0^2}$$

where χ is a function satisfying the equation (1.8).

We see that $w_0 = \chi$, i.e. the flexure of the middle plane also satisfies the equation (1.8). Thus we see that by performing the above substitution in (1.5) we arrive at the formulas expressing the displacement components and the function p in terms of the flexure of the middle plane

$$\begin{aligned} u_i &= -h\gamma\zeta\partial_i w_0 + h^3\gamma\zeta A_0(\zeta)\partial_i D^2 w_0 \qquad (i = 1, 2) \\ w &= w_0 + h^2 B_0(\zeta)\, D^2 w_0, \quad p = h\lambda^{-2}\, C_0(\zeta)\, D^2 w_0 \end{aligned} \tag{3.2}$$

$$A_0(\zeta) = \alpha_0^{-2} L(\gamma, \alpha_0)\,[(1+\gamma^2)\,R(\alpha_0\gamma\zeta, 1)\cos\alpha_0 - 2R(\alpha_0\zeta, \gamma)\cos\alpha_0\gamma]$$
$$B_0(\zeta) = \alpha_0^{-2} L(\gamma, \alpha_0)\,[2\gamma^2\cos\alpha_0\gamma\,(1-\cos\alpha_0\zeta) - (1+\gamma^2)\cos\alpha_0 \times (1-\cos\alpha_0\gamma\zeta)]$$
$$C_0(\zeta) = (\gamma\alpha_0)^{-1}(\gamma^4-1)\cos\alpha_0\, L(\gamma,\alpha_0)\sin\alpha_0\gamma\zeta$$
$$L(\gamma,\alpha_0) = [(1+\gamma^2)\cos\alpha_0 - 2\gamma^2\cos\alpha_0\gamma]^{-1}$$
$$R(x, y) = y^2 + x^{-1}\sin x$$

It can be confirmed that as $\gamma \to 1$, the relations (3.2) are transformed into the corresponding formulas of [2].

Using now (1.1), we write (3.2) in terms of the coordinates of the initial deformed state y_k. We obtain

$$u_i = -h\zeta_*\frac{\partial w_0}{\partial y_i} + h^3\lambda^2\zeta_* A_0(\lambda^2\zeta_*)\frac{\partial}{\partial y_i}\Delta w_0, \quad i = 1, 2 \tag{3.3}$$
$$w = w_0 + h^2 B_0(\lambda^2\zeta_*)\,\lambda^2\Delta w_0, \quad p = C_0(\lambda^2\zeta_*)\Delta w_0, \quad \zeta_* = y_3/h$$

where Δ denotes a two-dimensional Laplace operator in y_1 and y_2. The formulas (3.3) show that the lower order terms in h in the expression for the displacements correspond to the Kirchhoff's kinematic hypotheses appearing in the metric of the initial deformed state.

Let us now compare the equation of flexure of the middle plane

$$D^4 w_0 - \frac{\alpha_0^2}{h^2} D^2 w_0 = 0 \tag{3.4}$$

with the Saint Venant equation of the classical theory of stability of a plate [10]. Replacing in (1.4) λ by $\gamma^{-1/2}$ and expanding the right hand side into a series in $1-\gamma^2$, we obtain

$$\frac{\sigma}{G} = (1-\gamma^2)\left[1 + \frac{1}{3}(1-\gamma^2) + \frac{2}{9}(1-\gamma^2)^2 + \frac{14}{81}(1-\gamma^2)^3 + \ldots\right]$$

from which follows

$$1-\gamma^2 = \frac{\sigma}{G} - \frac{1}{3}\frac{\sigma^2}{G^2} + \frac{5}{27}\frac{\sigma^4}{G^4} + \ldots$$

Using (2.8) we find

$$\alpha_0^2 = \frac{3}{4}\frac{\sigma}{G} + \frac{19}{80}\frac{\sigma^2}{G^2} + \frac{107}{2800}\frac{\sigma^3}{G^3} + \ldots \tag{3.5}$$

and substituting (3.5) into (3.4) we obtain

$$\frac{1}{3} G(2h)^3 D^4 w_0 - 2h\sigma D^2 w_0 - \frac{hG}{30}\left(19\frac{\sigma^2}{G^2} + \frac{107}{35}\frac{\sigma^3}{G^3} + \ldots\right) D^2 w_0 = 0$$

For a plate made of an incompressible material, the cylindrical rigidity is equal to $G\,(2h)^3\,/\,3$. Consequently the classical Saint Venant equation holds with the accuracy of up to terms of order $(\sigma\,/\,G)^2$.

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